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A note on disjunctive Rado numbers

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Abstract

For all integers m, n, such that $3 \leqslant m \leqslant n$, let $r(S(m), S(n))$ represent the least integer such that for every 2-coloring of the set $\{1, 2, \ldots, r(S(m), S(n))\}$ there exists a monochromatic solution to either $S(m)$: $\sum_{i=1}^{m-1} x_i = x_m$ or $S(n)$: $\sum_{i=1}^{n-1} x_i = x_n$. The integer $r(S(m), S(n))$ is called the disjunctive Rado number for the above two equations. In this paper it is determined that

$$r\bigl(S(m), S(n)\bigr) = \begin{cases} m^2 - m - 1 & \text{for } m \leqslant n \leqslant m+1, \\ m^2 - 2m + 1 & \text{for } m+2 \leqslant n \leqslant m^2 - 2m + 2, \\ n - 1 & \text{for } m^2 - 2m + 3 \leqslant n \leqslant m^2 - m - 1, \\ m^2 - m - 1 & \text{for } n \geqslant m^2 - m. \end{cases}$$

Keywords: Rado numbers

1. Introduction

A function $\Delta : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, t-1\}$ is called a t-coloring of the set $\{1, 2, \ldots, n\}$. Given a coloring Δ and a system of equations in m variables, a monochromatic solution to the system is any solution $(x_1, x_2, \ldots, x_m)$, such that $\Delta(x_i) = \text{const}$ for $i = 1, 2, \ldots, n$.

It was proved by I. Schur [10] in 1916, that for every t-coloring of the natural numbers with t colors there exists a monochromatic solution to $x_1 + x_2 = x_3$. For a given natural number t, the least integer $n = S(t)$, such that for every t-coloring of the set $\{1, 2, \ldots, n\}$,

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there exists a monochromatic solution to this equation is called the t-color *Schur number.* The t-color Schur numbers are known only for a few small values of t [11].

In 1933, R. Rado, who was a student of Schur, determined necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution for every coloring of the natural numbers with a finite number of colors [7]. If L is a given system of linear equations, then the least integer $r(L)$, if it exists, such that for every coloring of the set $\{1, 2, \ldots, r(L)\}$ with t colors there exists a monochromatic solution to the system is called the t-color *Rado number* for the system. If such an integer does not exist, then the t-color Rado number for the system is infinite. Rado numbers are also referred to as generalized Schur numbers. Recently, the exact values of some simple one-equation Rado numbers have been found [3–5,9]. Two variations of the classical Rado numbers for a set of two equations have recently been introduced. The off-diagonal Rado numbers, which are similar to the classical off-diagonal Ramsey numbers [2], were introduced in [8] and are defined below.

Definition 1.1. Let L_0 and L_1 be equations. The *off-diagonal Rado number* $n = r_{\mathrm{od}}(L_0, L_1)$ is the least integer, if it exists, such that for every coloring of the set $\{1, 2, \ldots, n\}$ there exists a solution to L_0 which is monochromatic in 0 or there exists a solution L_1 which is monochromatic in 1. If such an integer does not exist, then the off-diagonal Rado number is infinite.

In this paper we consider the equation:

$$\sum_{i=1}^{m-1} x_i = x_m, \tag{$S(m)$}$$

where $x_1, x_2, \ldots, x_m \in \mathbb{N}$, $m \geqslant 3$. In 1982, A. Beutelspacher and W. Brestovansky [1] proved that the 2-color Rado number for $(S(m))$ is $m^2 - m - 1$. In [8] the set of two equations $(S(m))$ and $(S(n))$ was considered and it was determined that for all integers m, n such that $3 \leqslant m \leqslant n$,

$$r_{\mathrm{od}}\big(S(m), S(n)\big) = \begin{cases} 3n-4 & \text{if } m = 3 \text{ and } n \text{ is odd}, \\ 3n-5 & \text{if } m = 3 \text{ and } n \text{ is even}, \\ mn-n-1 & \text{if } 4 \leqslant m \leqslant n. \end{cases}$$

Another variation of the classical Rado numbers for a two-equation set of equations was introduced in [6].

Definition 1.2. Let L_0 and L_1 be equations. The *disjunctive Rado number* $n = r_d(L_0, L_1)$ is the least integer, if it exists, such that for every coloring of the set $\{1, 2, \ldots, n\}$ there exists a monochromatic solution to L_0 or there exists a monochromatic solution to L_1. If such an integer does not exist, then the disjunctive Rado number is infinite.

The following is the main result of this paper.

Theorem 1.1. *Let $3 \leqslant m \leqslant n$ be integers. The disjunctive Rado number for the equations $(S(m))$ and $(S(n))$ is*

$$r_d\big(S(m), S(n)\big) = \begin{cases} m^2 - m - 1 & \text{for } m \leqslant n \leqslant m+1, \\ m^2 - 2m + 1 & \text{for } m+2 \leqslant n \leqslant m^2 - 2m + 2, \\ n - 1 & \text{for } m^2 - 2m + 3 \leqslant n \leqslant m^2 - m - 1, \\ m^2 - m - 1 & \text{for } n \geqslant m^2 - m. \end{cases}$$

There are two obvious relationships between various Rado numbers for a given set of equations L_0 and L_1. It is clear that

$$r_d(L_0, L_1) \leqslant \min\big\{r(L_0), r(L_1)\big\}.$$

It follows immediately from Theorem 1.1 and [1], that

$$r_d\big(S(m), S(n)\big) = \min\big\{r\big(S(m)\big), r\big(S(n)\big)\big\}$$

if and only if $n = m + 1$ or $n \geqslant m^2 - m$ (and of course trivially if $n = m$).

It is also clear that for any set consisting of the two equations L_0 and L_1

$$r_d(L_0, L_1) \leqslant r_{\mathrm{od}}(L_0, L_1).$$

Again, it follows from Theorem 1.1 and [8], that for all integers $3 \leqslant m < n$ we have

$$r_d\big(S(m), S(n)\big) < r_{\mathrm{od}}\big(S(m), S(n)\big).$$

It would be interesting to find out under what conditions the disjunctive Rado numbers and the off-diagonal Rado numbers for a two-equation set are the same (and finite). There certainly exist some simple sets like that, for example, the set consisting of equations $x_1 + x_2 - 1 = x_3$ and $2x_1 + x_2 - 2 = x_3$, for which both the off-diagonal and the disjunctive Rado numbers are 1. What happens in the case of less trivial sets is still unknown.

2. Proof

Without loss of generality we will assume throughout this paper that $\Delta(1) = 0$ for all colorings considered. We will also say that a 2-coloring $\Delta : \{1, 2, \ldots, n\} \to \{0, 1\}$ is *m-good* if it avoids a monochromatic solution to $(S(m))$.

Lemma 2.1. *Every m-good coloring $\Delta : \{1, 2, \ldots, (m-1)^2\} \to \{0, 1\}$ satisfies the property*:

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \leqslant x \leqslant m-2 \text{ or } x = (m-1)^2, \\ 1 & \text{for } m-1 \leqslant x \leqslant m^2 - 3m + 3. \end{cases} \tag{1}$$

Proof. Suppose Δ is a good coloring of $\{1, 2, \ldots, (m-1)^2\}$. Since

$$x_1 = \cdots = x_{m-1} = 1, \qquad x_m = m-1$$

is a solution to $(S(m))$, we have $\Delta(m-1) = 1$. Also, since

$$x_1 = \cdots = x_{m-1} = m-1, \qquad x_m = (m-1)^2$$

is a solution to $(S(m))$, we have $\Delta((m-1)^2) = 0$.

Suppose that $\Delta(m-2) = 1$. We would then have $\Delta(x) = 0$ for all $(m-1)(m-2) \leqslant x \leqslant (m-1)^2$ because

$$x_1 = \cdots = x_k = m-1, \qquad x_{k+1} = \cdots = x_{m-1} = m-2, \qquad x_m = (m-1)(m-2) + k$$

is a solution to $(S(m))$, for every $k = 0, 1, \ldots, m-1$. But then

$$x_1 = \cdots = x_{m-2} = 1, \qquad x_{m-1} = (m-1)(m-2) + 1, \qquad x_m = (m-1)^2$$

would be a monochromatic solution to $(S(m))$, a contradiction. Hence $\Delta(m-2) = 0$.

We will show by induction that $\Delta(m-2-k) = 0$, for $k = 1, 2, \ldots, m-3$.

For $k = 1$, since

$$x_1 = 1, \qquad x_2 = \cdots = x_{m-1} = m-2, \qquad x_m = m^2 - 4m + 5$$

is a solution to $(S(m))$, we have $\Delta(m^2 - 4m + 5) = 1$. Suppose that $\Delta(m-3) = 1$. Then

$$x_1 = \cdots = x_{m-2} = m-3, \qquad x_{m-1} = m-1, \qquad x_m = m^2 - 4m + 5$$

is a monochromatic solution to $(S(m))$, a contradiction.

Induction step. Assume that $\Delta(m-2-i) = 0$, for $i = 0, 2, \ldots, k$, $1 \leqslant k \leqslant m-4$. It is easy to see that linear combinations of integers from the interval $[m-2-k, m-2]$ taken as a left-hand side of $(S(m))$ cover all integers in the interval $I = [(m-1)(m-2-k), (m-1)(m-2)]$. It follows that $\Delta(x) = 1$ for all $x \in I$.

Suppose now that $\Delta(m-3-k) = 1$. Let $j \in \{0, 1, \ldots, m-1\}$. Consider

$$x_1 = \cdots = x_j = m-1, \qquad x_{j+1} = \cdots = x_{m-1} = m-3-k.$$

Let

$$x_m^{(j)} = \sum_{i=1}^{m-1} x_i = j(m-1) + (m-1-j)(m-3-k) = (m-1)(m-3-k) + j(k+2).$$

Then $x_m^{(0)} < (m-1)(m-2-k)$ and $x_m^{(m-1)} > (m-1)(m-2)$. Also, $\Delta(x_m^{(j)}) = 0$, for every $j \in \{0, 1, \ldots, m-1\}$. Since the length of the interval I is greater than $k+2$, there exists a $j \in \{0, 1, \ldots, m-1\}$, such that $x_m^{(j)} \in I$, a contradiction.

We have thus shown that $\Delta(x) = 0$ for $1 \leqslant x \leqslant m-2$ or $x = (m-1)^2$. As an immediate consequence of this we obtain that $\Delta(x) = 1$ for $m-1 \leqslant x \leqslant (m-1)(m-2)$.

Finally, since

$$x_1 = \cdots = x_{m-2} = 1, \qquad x_{m-1} = m^2 - 3m + 3, \qquad x_m = (m-1)^2$$

is a solution to $(S(m))$, and $\Delta(x_1) = \cdots = \Delta(x_{m-2}) = \Delta(x_m) = 0$, we have $\Delta(m^2 - 3m + 3) = 1$. $\square$

Remark 2.1. It is easy to verify that the property (1) characterizes all the m-good colorings of the interval $[1, (m-1)^2]$. In other words, one can color all the numbers in the interval $[m^2 - 3m + 4, m^2 - 2m]$ in an arbitrary fashion and the coloring will be m-good, as long as the property (1) is satisfied.

The following is a straightforward generalization of Lemma 2.1:

Proposition 2.1. *For every* $p = 0, 1, \ldots, m-3$, *every* m*-good coloring* $\Delta : \{1, 2, \ldots, m - 2m + 1 + p\} \to \{0, 1\}$ *satisfies the property*:

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \leqslant x \leqslant m-2, \\ 1 & \text{for } m-1 \leqslant x \leqslant m^2 - 3m + 3 + p, \\ 0 & \text{for } m^2 - 2m + 1 \leqslant x \leqslant m^2 - 2m + 1 + p. \end{cases} \tag{2}$$

Proof. Every m-good coloring Δ of the interval $[1, (m-1)^2 + p]$ is also an m-good coloring of $\{1, \ldots, (m-1)^2\}$. It follows from Lemma 2.1 that we only need to check that $\Delta(x) = 1$ for $m^2 - 3m + 3 < x \leqslant m^2 - 3m + 3 + p$ and $\Delta(x) = 0$ for $(m-1)^2 < x \leqslant (m-1)^2 + p$. For every $j = 0, 1, \ldots, p$ let us consider the following solution to $(S(m))$:

$$x_1 = \cdots = x_j = m, \qquad x_{j+1} = \cdots = x_{m-1} = m - 1, \qquad x_m = (m-1)^2 + j.$$

Since $\Delta(x_1) = \cdots = \Delta(x_{m-1}) = 1$, we must have $\Delta((m-1)^2 + j) = 0$.

Next, for every $j = 0, 1, \ldots, p$ take

$$x_1 = \cdots = x_{m-2} = 1, \qquad x_{m-1} = m^2 - 3m + 3 + j, \qquad x_m = (m-1)^2 + j.$$

Again, since $\Delta(x_1) = \cdots = \Delta(x_{m-2}) = \Delta(x_m) = 0$, we have $\Delta(x_m) = 1$. $\square$

Remark 2.2. The number of m-good colorings steadily decreases with increasing values of p, coming down to just one m-good coloring for $p = m - 3$. Since $(m-1)^2 + m - 2$ is the Rado number for $(S(m))$, for $p = m - 2$, there are no m-good colorings of $\{1, \ldots, (m-1)^2 + m - 2\}$.

Proof of Theorem 1.1. For $n = m$ the result follows trivially from the fact that $r(S(m))$ is $m^2 - m - 1$ (as shown in [1]). For all values of n the upper bound for $r_d(S(m), S(n))$ is also $m^2 - m - 1$.

For $n = m + 1$, the same coloring which avoids the monochromatic solution to $(S(m))$ also avoids the solution to $(S(m+1))$. In fact, consider $\Delta : \{1, 2, \ldots, m^2 - m - 2\} \to \{0, 1\}$ defined as

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \leqslant x \leqslant m - 2, \\ 1 & \text{for } m - 1 \leqslant x \leqslant m^2 - 2m, \\ 0 & \text{for } m - 2m + 1 \leqslant x \leqslant m^2 - m - 2. \end{cases}$$

Suppose that $x_1, x_2, \ldots, x_{m+1}$ solve the equation $(S(m + 1))$. Suppose that $\Delta(x_i) = 0$ for all $i = 1, 2, \ldots, m$. If $x_1, x_2, \ldots, x_m \leqslant m - 2$. Then $m \leqslant x_{m+1} \leqslant m(m - 2) = m^2 - 2m$ and hence $\Delta(x_{m+1}) = 1$. If $x_i > (m - 1)^2$ for some $i \in \{1, 2, \ldots, m\}$, then $x_{m+1} > m - 1 + (m - 1)^2 = m^2 - m$, a contradiction. Next, suppose $\Delta(x_i) = 1$ for all $i = 1, 2, \ldots, m$. Then $x_m \geqslant m(m - 1) \geqslant (m - 1)^2$ and $\Delta(x_m) = 0$. Therefore, Δ is an $(m + 1)$-good coloring.

Suppose now that $m + 2 \leqslant n \leqslant m^2 - 2m + 2$. For the upper bound let us suppose that Δ is a coloring of $\{1, 2, \ldots, m^2 - 2m + 1\}$ which is both m-good and n-good. By Lemma 2.1 the coloring Δ satisfies the property (1). In particular we have $\Delta(1) = \Delta(m - 2) = 0$. It is easy to see that by looking at all possible choices of the numbers $x_1, x_2, \ldots, x_{n-1}$ from the set $\{1, 2, \ldots, m - 2\}$ one can be sure that $\Delta(x) = 1$ for all integers $x \in [n - 1, (n - 1) \times (m - 2)]$. Since $n - 1 \leqslant m^2 - 2m + 1$ and $(n - 1)(m - 2) \geqslant (m + 2)(m - 2) > m^2 - 2m + 1$, we have a contradiction with $\Delta(m^2 - 2m + 1) = 0$.

On the other hand, the coloring $\Delta : \{1, 2, \ldots, m^2 - 2m\} \to \{0, 1\}$ defined by

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \leqslant x \leqslant m - 2, \\ 1 & \text{for } m - 1 \leqslant x \leqslant m^2 - 2m \end{cases}$$

is obviously both m-good and n-good.

Next, suppose that $n = m^2 - 2m + 2 + p$, for some $p \in \{1, 2, \ldots, m - 3\}$. Again, for the upper bound let us suppose that Δ is a coloring of $\{1, 2, \ldots, n - 1\}$ which is both m-good and n-good. By Proposition 2.1 the coloring Δ satisfies the property (2). In particular, $\Delta(1) = \Delta(n - 1) = 0$. By taking $x_1 = x_2 = \cdots = x_{n-1} = 1$, and $x_n = n - 1$, we would have a monochromatic solution, a contradiction.

On the other hand, the coloring $\Delta : \{1, 2, \ldots, n - 2\} \to \{0, 1\}$ defined by

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \leqslant x \leqslant m - 2, \\ 1 & \text{for } m - 1 \leqslant x \leqslant m^2 - 2m, \\ 0 & \text{for } m - 2m + 1 \leqslant x \leqslant n - 2 \end{cases}$$

is both m-good and n-good.

Finally, suppose that $n \geqslant m^2 - m$. In this case the disjunctive Rado number $r_d(S(m), S(n))$ is the same as the regular Rado number $r(S(m))$, simply because there are no solutions to $(S(n))$, monochromatic or not, with $x_1, \ldots, x_n \in \{1, 2, \ldots, m^2 - m - 2\}$. $\square$

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